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LOCAL FIXED POINT INDEX THEORY FOR NON SIMPLY CONNECTED MANIFOL--ETC(U)
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LOCAL FIXED POINT INDEX THEORY FOR NON SIMPLY CONNECTED MANIFOLDS

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LOCAL FIXED POINT INDEX THEORY FOR NON SIMPLY CONNECTED MANIFOLDS

Edward Fadell[†] and Sufian Husseini[†]

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ABSTRACT

Given a compactly fixed map $f: U \to M$, where U is an open subset of a manifold M, it is a classical result that one can assign an integer-valued index $I(f,U)$ to this situation with the property that $I(f,U) \neq 0$ implies f (and any compactly fixed perturbation of f) has fixed points in U, i.e. solutions to the equations $f(x) = x$. However, it can still happen that $I(f,U) = 0$ and f has essential fixed points in U. The objective of this paper is to provide a finer invariant $o(f,U)$, called the <u>local obstruction index</u>, which has the property that $o(f,U) \neq 0$ <u>if, and only if</u>, every compactly fixed perturbation of f has fixed points in U. $o(f,U)$ is not integer-valued but takes its value initially in the cohomology group $H_c^m(U, \mathcal{B}(f))$ where $m = \dim M$, and $\mathcal{B}(f)$ is an appropriate local coefficient system on U with local group $\mathbb{Z}[\pi]$, $\pi = \pi_1(M)$. In order to compute $o(f,U)$ one employs a Kronecker product

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$$\langle \cdot , \mu \rangle : H_c^m(U;\mathcal{B}(f)) \to \mathbb{Z}R[i_U, \varphi_U] \quad .$$

$R[i_U, \varphi_U]$ is the set of Reidemeister classes obtained from the group π using the Reidemeister action

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$$\alpha * \sigma = \varphi_U(\sigma^{-1})\alpha\, i_U(\sigma) \quad , \quad \sigma \in \pi(U) = \pi_1(U),\ \alpha \in \pi$$

where $i_U : \pi(U) \to \pi$ is induced by inclusion, $\varphi_U = f_* : \pi(U) \to \pi$, and μ is the twisted fundamental class on U. Then, we have the following basic result:

<u>Theorem</u>. $\langle o(f,U), \mu\rangle = \sum_{\rho \in R} I(f,\rho)\rho$

where $R = R[i_U, \varphi_U]$, and $I(f,\rho)$ is the classical integer-valued index of f on the Nielsen class corresponding to the Reidemeister class ρ.

AMS (MOS) Subject Classifications: 54H25, 55C20, 55G30

Key Words: Local index theory, local Nielsen theory, local obstruction index, local Reidemeister classes.

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SIGNIFICANCE AND EXPLANATION

An important problem in mathematics and applied areas is to find solutions to an equation of the form $f(x) = x$ on a given region or manifold M. Such solutions are referred to as "fixed points". Here f is a given formula (continuous mapping) on the manifold. It is also convenient to determine whether solutions to the same equation exist in a given subregion U of the manifold. One classical approach to this problem is through "Fixed Point Index Theory". Specifically one assigns to the data (f,U) an integer $I(f,U)$ with the property that when $I(f,U) \neq 0$, then solutions to the equation $f(x) = x$ exist in U and furthermore, solutions survive under perturbations of the data (essential solutions). One can think of this index $I(f,x)$ as a "local degree" similar to the concept of "winding number" in complex variable theory. However, on complicated manifolds it is possible that $I(f,U) = 0$, and essential solutions still exist. This ~~is a considerable defect of this index~~ cont. The main objective ~~of this paper~~ is to assign to this data a new index, $o(f,U)$, which is not necessarily an integer, which removes this fundamental defect so that now the equation has essential solutions in U. ~~if, and only if, $o(f,U) \neq 0$.~~ A general formula for computing the value of $o(f,U)$ is also given.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

LOCAL FIXED POINT INDEX THEORY
FOR NON SIMPLY CONNECTED MANIFOLDS

Edward Fadell† and Suffian Husseini†

1. Introduction

This paper is a sequel to [1]. There we associated to a globally defined map $f : M \to M$ on a compact manifold an obstruction class $o(f) \in H^m(M;\mathcal{B}(f))$, $m = \dim M$, where $\mathcal{B}(f)$ is an appropriate bundle of groups on M, with local group isomorphic to $\mathbf{Z}[\pi]$, $\pi = \pi_1(M)$. We also identified $o(f)$ with an element $L_\pi(f) \in \mathbf{Z}R[\pi,\varphi]$, where $R(\pi,\varphi)$ is the set of Reidemeister classes of π induced by the homomorphism $\varphi = f_\# : \pi \to \pi$. $L_\pi(f)$ had the form

$$L_\pi(f) = \pm \sum_{\rho\in R} I(\rho)\rho$$

where $R = R[\pi,\varphi]$ and $I(\rho)$ is the index of the Nielsen class of f corresponding to ρ. This gave us a specific relationship between the obstruction $o(f)$ and the Nielsen number $n(f)$ of f, or, more precisely, between $o(f)$ and a <u>generalized Lefschetz number</u> $L_\pi(f)$ which played the role of a global index and which, in turn, was expressible in terms the Nielsen classes of f. As a consequence, for example, $L_\pi(f) = 0$ forces $o(f) = 0$ and one obtains the appropriate converse of the Lefschetz Fixed Point Theorem for non-simply connected manifolds.

Our objective here is to carry out this program locally and thereby give a generalized local index theory.

Section 2 is devoted to the local obstruction index. Starting with a smooth or PL manifold M, $\dim M \geq 3$, the inclusion map $M \times M - \Delta \hookrightarrow M \times M$ is replaced by a fiber map $p : E \to M \times M$ and the bundle $\mathcal{B}$ of coefficients is the local system $\pi_{m-1}(F)$ on $M \times M$, where F is the fiber of p. The group $\pi_{m-1}(F)$ is identified in [1] as $\mathbf{Z}[\pi]$, where $\pi = \pi_1(M)$ and the action of $\pi \times \pi$ on $\mathbf{Z}[\pi]$ is given by the right action

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$$\alpha \circ (\sigma,\tau) = (\operatorname{sgn} \sigma)\sigma^{-1}\alpha\,\tau \; .$$

Now, we suppose that we are given a map $f : U \to M$, which is compactly fixed on U (i.e. Fix f is compact), U an open set in M. Let $\mathcal{B}(f)$ denote the bundle of groups on M induced from $\mathcal{B}$ by $i \times f : U \to M \times M$. The local obstruction index

$$o(f) = o(f,U) \in H_c^m(U;\mathcal{B}(f))$$

is defined by first taking a compact m-manifold K with boundary ∂K such that $K \subset U$ and $(\mathrm{Fix}\ f) \subset \mathrm{int}\ K$. Then, if $E(f)$ is the induced fiber space $(i \times f)^*(E)$, there is a natural partial section $s_o(f) : \partial K \to E(f)$ and, consequently, a primary obstruction

$$o(f,K) \in H^m(K,\partial K;\mathcal{B}(f,K))$$

with the property that f is deformable (rel ∂K) to a fixed point free map (into M) if, and only if, $o(f,K) = 0$. By letting C denote a slightly smaller copy of K, $o(f,K)$ determines an element of $H^m(U, U-C)$ and consequently the element

$$o(f) \in H_c^m(U;\mathcal{B}(f))$$

called the local obstruction index of f on U. Among others, it has the property that f can be deformed by a compactly fixed homotopy to a fixed point free map g if, and only if, $o(f) = 0$.

In section 3 we study local Nielsen numbers in a more general situation. Here $f : U \to X$ is a compactly fixed map and X is a Euclidean neighborhood retract (ENR [2]). Given two points x_1 and x_2 in Fix f we say that x_1 and x_2 are Nielsen equivalent if there is a path C in U from x_1 to x_2 such that C and Cf are homotopic in X, modulo end-points. The resulting classes (finite in number) are called Nielsen classes of f in U. Such a Nielsen class $N(f,U)$ is essential if the local (numerical) index [2] is not zero on $N(f,U)$. The local Nielsen number $n(f,U)$ on U is just the number of such essential classes. We also express the local Nielsen classes in terms of the universal covers $\eta_U : \tilde{U} \to U$, $\eta : \tilde{X} \to X$. One takes lifts $\tilde{i} : \tilde{U} \to \tilde{X}$, $\tilde{f} : \tilde{U} \to \tilde{X}$ of the inclusion i and the map f and identifies π and $\pi(U)$ with the covering groups of η and η_U, respectively. Then, a typical Nielsen class has the form

$$\eta_U(\mathrm{Coin}[\tilde{f}\alpha, \tilde{i}]), \quad \alpha \in \pi \quad .$$

where $\mathrm{Coin}[\cdot,\cdot]$ is the coincidence set of two maps. Next, we employ the notion of Reidemeister classes in the situation of two homomorphisms

$$\psi : \pi' \to \pi \qquad \varphi : \pi' \to \pi$$

which induces the right π'-action on π by $\alpha * \sigma = \varphi(\sigma^{-1})\alpha\psi(\sigma)$. The resulting set of orbits (Reidemeister classes) is denoted by $R[\psi, \varphi]$. The relationship between local Nielsen classes and Reidemeister classes is as follows: Let $i_U : \pi(U) \to \pi$, $\varphi_U : \pi(U) \to \pi$ denote the homomorphisms induced by the inclusion and the map f. The correspondence $\Gamma : [\alpha] \mapsto \eta_U(\mathrm{Coin}[\tilde{f}\alpha, \tilde{i}])$ takes $R[i_U, \varphi_U]$ bijectively to the set of Nielsen classes of f on U, if we ignore those Reidemeister classes for which $\eta_U(\mathrm{Coin}[\tilde{f}\alpha, \tilde{i}]) = \phi$. Using the correspondence Γ the index $I(\rho)$ of a Reidemeister class $\rho \in R[i_U, \varphi_U]$ is defined to be the index of the corresponding Nielsen class $\Gamma(\rho)$.

In order to calculate the local obstruction index $o(f)$ when U is connected, (§§4, 5) we make use of a bilinear pairing of local systems

$$P : \mathcal{B}(f) \otimes \mathcal{T}(U) \to \mathcal{R}(f)$$

where $\mathcal{T}(U)$ is the orientation sheaf on U and $\mathcal{R}(f)$ is the local system on U with local group $\mathbb{Z}[\pi]$ and action

$$\alpha * \sigma = \varphi_U(\sigma^{-1})\alpha \; i_U(\sigma) \quad .$$

Then, if $\underline{\mu}(U) \in H_m^c(U; \mathcal{T}(U))$ is the twisted fundamental class on U we have a cap product based on the above pairing and a Kronecker product

$$\langle \cdot, \underline{\mu}(U) \rangle : H_c^m(U; \mathcal{B}(f)) \to H_o(U; \mathcal{R}(f)) \equiv \mathbb{Z}R[i_U, \varphi_U] \quad .$$

We are now in a position to state the main theorem which expresses the local obstruction index $o(f)$ in terms of Reidemeister (Nielsen) class of f on U.

<u>Theorem</u>. Let $R = R[i_U, \varphi_U]$. Then

$$\langle o(f), \underline{\mu}(U) \rangle = (-1)^m \sum_{\rho \in R} I(\rho)\rho \in \mathbb{Z}R[i_U, \varphi_U] \quad .$$

Corollary. $f : U \to M$ is deformable via a compactly fixed homotopy to a fixed point free map $g : U \to M$ if, and only if, the local Nielsen number $n(f,U) = 0$.

2. The Local Obstruction

Let M denote a connected (not necessarily compact) manifold of dimension $m \geq 3$, and $\Delta_M = \Delta \subset M \times M$ the diagonal. Then, if we replace the inclusion map $i : M \times M - \Delta \subset M \times M$ by a fiber map $p : E \to M \times M$, we recall [1] that

$$E = \{(\alpha,\beta) \in M^I \times M^I : \alpha(0) \neq \beta(0)\}$$

where I is the interval $[0,1]$ and $p(\alpha,\beta) = (\alpha(1),\beta(1))$. Furthermore, if $b = (x,y) \in M \times M$, the fiber

$$F_b = p^{-1}(b) = \{(\alpha,\beta) \quad E : \alpha(1) = x,\ \beta(1) = y\}$$

is 1-connected, so that F_b is k-simple for every k and $\pi_{m-1}(F_b)$ is a bundle (local system) of groups on $M \times M$. We denote this bundle by $\mathcal{B} = \mathcal{B}(M \times M)$. In [1], we obtained a description of the structure of $\mathcal{B}$ as follows: We fix a base point $b = (x,y) \in M \times M - \Delta$ and let $\bar{b}$ denote the constant path at b. Then we identify π with $\pi_1(M,x)$ and $\pi \times \pi$ with $\pi_1(M,x) \times \pi_1(M,y)$, with x near, but distinct from, y. Then, there is an isomorphism of local systems (on $M \times M - \Delta$)

$$\psi : \pi_m(M \times M, M \times M - \Delta, b) \to \pi_{m-1}(F_b,\bar{b})$$

given by the exponential map and ψ was employed to establish the following theorem.

2.1 <u>Theorem</u>. There is an equivariant isomorphism

$$\xi : \mathbb{Z}[\pi] \to \pi_{m-1}(F_b,\bar{b})$$

where the action of $\pi \times \pi$ on $\pi_{m-1}(F_b,\bar{b})$ is given by $\mathcal{B}$ and the action of $\pi \times \pi$ on $\mathbb{Z}[\pi]$ is given by the right action

$$\alpha \circ (\sigma,\tau) = (\operatorname{sgn} \sigma)\sigma^{-1}\alpha\,\tau \quad .$$

σ and τ belong to π and $\operatorname{sgn} \sigma$ is ± 1 according as σ preserves or reverses a local orientation at $x \in M$.

2.2 <u>Remark</u>: If π is identified with covering transformations of $\eta : \tilde{M} \to M$, the universal cover of M, then $\sigma^{-1}\alpha\tau$ is to be read as composition of functions from left to right. In fact, we will, in general, write compositions of functions from left to right. However, we will still write $\alpha(x)$ for the value of the function α at x and thus we will also write, for example,

$$(\alpha\beta\gamma)(x) = \gamma(\beta(\alpha(x))) \quad .$$

In general group actions will be from the right and if π acts on X, $x\alpha$ may be used for the action of $\alpha \in \pi$ on $x \in X$ as well as $\alpha(x)$. In [1], we used the corresponding left action

$$(\sigma,\tau) \circ \alpha = (\mathrm{sgn}\ \sigma)\tau\ \alpha\ \sigma^{-1}$$

reading composition of functions from right to left.

We review briefly this isomorphism ξ in Theorem 2.1. ξ is obtained by establishing an isomorphism

$$\nu : \mathbf{Z}[\pi] \to \pi_m(M \times M, M \times M - \Delta, b)$$

and setting $\xi = \nu\psi$. The structure of ν is a bit involved and takes the following form.

Again, let $\eta : \tilde{M} \to M$ denote the universal cover of M. Choose a base point $\tilde{x}_1 \in \tilde{M}$ over x. We identify π with the covering group of η and if we set $\tilde{x}_\alpha = \tilde{x}_1\alpha$, $\alpha \in \pi$, then $\eta^{-1}(x) = \{\tilde{x}_\alpha, \alpha \in \pi\}$. The diagram

$$(1) \qquad \begin{array}{ccc} \tilde{M} & \xleftarrow{\mathrm{proj}_1} & (\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta)) \\ \eta \downarrow & & \downarrow \zeta \\ M & \xleftarrow{\mathrm{proj}_1} & (M \times M, M \times M - \Delta) \end{array}$$

where $\zeta = \eta \times \eta$ and the horizontal maps of (1) are fibered pair projections on the first coordinate, gives rise to isomorphisms for each σ, τ.

$$(2) \qquad \begin{array}{ccc} \pi_m(\tilde{M}, \tilde{M} - y^{-1}(x), \tilde{y}_\tau) & \xrightarrow{\approx} & \pi_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta), (\tilde{x}_\sigma, \tilde{y}_\tau)) \\ \approx \downarrow & & \approx \downarrow \\ \pi_m(M, M - x, y) & \xrightarrow{\approx} & \pi_m(M \times M, M \times M - \Delta, (x, y)) \end{array}$$

where $(M, M - x)$ and $(\tilde{M}, \tilde{M} - \eta^{-1}(x))$ are the fiber pairs of the horizontal maps in (1). In (2), $\tilde{y}_\tau = \tau\tilde{y}_1$, where $\tilde{y}_1$ lies over y and $\tilde{y}_1$ is chosen near $\tilde{x}_1$. Also, the top horizontal isomorphism in (2) is induced by the fiber inclusion

$$\theta_\sigma : (\tilde{M}, \tilde{M} - \eta^{-1}(x)) \subset (\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta))$$

given by $\theta_\sigma(u) = (\tilde{x}_\sigma, u)$. Applying the Hurewicz Isomorphism Theorem, we have

$$\begin{tikzcd}
\pi_m(\tilde{M}, \tilde{M} - \eta^{-1}(x), \tilde{y}_\tau) \arrow[r, "\theta_{\sigma\#}"] \arrow[d, "\approx"'] & \pi_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta), (\tilde{x}_\sigma, \tilde{x}_\tau)) \arrow[d, "\approx"] \\
H_m(\tilde{M}, \tilde{M} - \eta^{-1}(x)) \arrow[r, "\theta_{\sigma*}"] & H_m(\tilde{M} \times \tilde{M}, \tilde{M} - \zeta^{-1}(\Delta))
\end{tikzcd} .$$

Now, choose a cell neighborhood V of x and corresponding neighborhoods $\tilde{V}_\alpha$ of $\tilde{x}_\alpha$, evenly covering V so that $\tilde{V}_1\alpha = \tilde{V}_\alpha$. Choose a local orientation at x, thereby determining a generator

$$\gamma_1 \in H_m(\tilde{V}_1, \tilde{V}_1 - \tilde{x}_1)$$

and since

$$H_m(\tilde{M}, \tilde{M} - \eta^{-1}(x)) \approx \sum_{\alpha \in \pi} H_m(\tilde{V}_\alpha, \tilde{V}_\alpha - x_\alpha) ,$$

the correspondences $\alpha \longmapsto \gamma_1\alpha \longmapsto \theta_{1*}(\gamma_1\alpha)$ give rise to the isomorphism ν as the following composition

$$\begin{tikzcd}
\mathbb{Z}[\pi] \arrow[r, "\approx"] & \sum_{\alpha\ \pi} H_m(\tilde{V}_\alpha, \tilde{V}_\alpha - \tilde{x}_\alpha) \arrow[r, "\approx"] & H_m(\tilde{M}, \tilde{M} - \eta^{-1}(x)) \arrow[d, "\theta_{1*}"] \\
& \pi_m(M \times M, M \times M - \Delta, b) & H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta) \arrow[l, "\approx"']
\end{tikzcd} .$$

This completes the sketch of the structure of ξ. While ξ does not depend on the choice for $\tilde{x}_1$ over x_1, ξ does depend on the orientation chosen at x and the choice of the base point $b = (x,y)$.

There is also an alternative description of ξ. Define a correspondence

$$\mu : \mathbb{Z}[\pi \times \pi] \longrightarrow H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta))$$

by setting

$$\mu(\alpha,\beta) = \theta_{1*}\gamma_1(\alpha \times \beta) \quad .$$

We factor out the subgroup D of $\mathbf{Z}[\pi \times \pi]$ generated by elements of the form

$$\operatorname{sgn} \sigma(\alpha\sigma,\beta\sigma) - (\alpha,\beta), \ \sigma, \ \alpha, \ \beta \in \pi \ .$$

Since [1], for every $\sigma \in \pi$

$$\theta_{1*}\gamma_1(\sigma \times \sigma) = (\operatorname{sgn} \sigma)\theta_{1*}\gamma_1$$

μ induces

$$\bar{\mu} : \mathbf{Z}[\pi \times \pi]/_D \longrightarrow H_m(\tilde{M} \times \tilde{M}, \ \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta)) \quad .$$

Now, let $\omega : \mathbf{Z}[\pi \times \pi] \longrightarrow \mathbf{Z}[\pi]$ be defined by

$$\omega(\alpha,\beta) = (\operatorname{sgn} \alpha)\alpha^{-1}\beta \quad .$$

Then, $\omega(D) = 0$, and we have an induced isomorphism

$$\bar{\omega} : \mathbf{Z}[\pi \times \pi]/_D \longrightarrow \mathbf{Z}[\pi] \quad .$$

Thus, ξ is also given by the following composition

$$\begin{tikzcd}
\mathbf{Z}[\pi] \arrow[dr] & \mathbf{Z}[\pi \times \pi]/_D \arrow[l, "\bar{\omega}"', "\approx"] \arrow[r, "\bar{\mu}", "\approx"'] & H_m(\tilde{M} \times \tilde{M}, \ \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta)) \arrow[dl, dashed, "\approx"] \\
& \pi_m(M \times M, \ M \times M - \Delta, \ b) &
\end{tikzcd}$$

and $\bar{\omega}$ and $\bar{\mu}$ are equivalent with respect to the right actions of $\pi \times \pi$ given respectively, when $(\sigma,\tau) \in \pi \times \pi$, by

$$\alpha(\sigma,\tau) = \operatorname{sgn} \sigma \ \sigma^{-1}\alpha \ \tau, \quad \alpha \in \pi \quad ,$$

$$[(\alpha,\beta)](\sigma,\tau) = [(\alpha\sigma,\beta\tau)], \quad (\alpha,\beta) \in \pi \times \pi \quad ,$$

$$u(\sigma,\tau) = (\sigma \times \tau)_*(u), \quad u \in H_m(\tilde{M} \times \tilde{M}, \ \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta)) \quad .$$

We now consider the following data which we designate by (M,f,U):

2.3 The data (M,f,U):

(i) M is a smooth or PL manifold of dimension $m \geq 3$.

(ii) U is an open subset of M.

(iii) $f : U \to M$ is a map with *compact* fixed point set $\mathrm{Fix}\ f \subset U$; i.e. f is <u>compactly fixed</u>.

This data is accompanied by the following ingredients with notation as follows:

(iv) $i : U \hookrightarrow M$, inclusion map

(v) $\mathcal{B}(f)$ the bundle of coefficients (local system) on U induced by $i \times f : U \to M \times M$ from $\mathcal{B} = \mathcal{B}(M \times M)$, i.e. $\mathcal{B}(f) = (i \times f)^*(\mathcal{B}(M \times M))$

(vi) $p_U : E(f) \to U$, the fiber space over U induced from $p : E \to M \times M$ by $i \times f$, i.e., $E(f) = (i \times f)^*(E)$.

Our objective is to define a local obstruction index $o(f) \in H^m_c(U,\mathcal{B}(f))$. To this end let K denote a triangulable compact m-manifold in U with boundary ∂K such that $(\mathrm{Fix}\ f) \cap \partial K = \phi$. Define a partial section $s_o(f) : \partial K \to E(f)$ by

$$s_o(f)(x) = (\bar{x}, \overline{f(x)})$$

where $\bar{u}$ denotes the constant path at u. Furthermore, let $\mathcal{B}(f,K)$ denote the restriction of $\mathcal{B}(f)$ to K.

2.4 <u>Lemma</u>. Let K be as above. Then, $f|K$ is deformable, relative to ∂K, to a map $g : K \to M$ which is fixed point free on K, iff, $s_o(f)$ admits an extension to a section over K.

Proof: The "only if" part is obvious. The "if part" requires a simple covering homotopy argument to adjust the section to have a constant path in the first coordinate [1].

2.5 <u>Definition.</u> Let $o(f,K) \in H^m(K,\partial K;\mathcal{B}(f,K))$ denote the primary obstruction to extending $s_o(f)$ to a section $s(f)$ over K. $o(f,K)$ will be called the <u>local obstruction index of f on K</u> $\subset U$.

General obstruction theory ([3]) implies the following proposition.

2.6 <u>Proposition</u>. $f|K$ is deformable to be fixed point free, relative to ∂K, iff the local obstruction index of f on K $o(f,K) = 0$.

Now let $\Gamma(U)$ denote the compact subsets C of U directed by inclusion and consider

$$H_c^m(U; \mathcal{E}(f)) = \varinjlim H^m(U, U - C; \mathcal{E}(f))$$

where the direct limit is over $\Gamma(U)$. Also suppose that $\operatorname{Fix} f \subset \operatorname{int} K$. The "excision" isomorphism,

$$H^m(U, U - K_o, \mathcal{E}(f)) \approx H^m(K, \partial K; \mathcal{E}(f,K)) \quad ,$$

where K_o is K minus a "collar" of ∂K, tells us that $o(f,K)$ determines an element $o(f) \in H_c^m(U; \mathcal{E}(f))$.

2.7 Definition-Proposition. $o(f)$ is independent of K and is called the local obstruction index of f.

Proof (of independence on K): Given K and K', choose K'' such that $K \cup K' \subset K''$. The diagram

$$\begin{tikzcd}
H^m(U, U - K''_o; \mathcal{E}(f)) \arrow[d, "\approx"'] & & H^m(U, U - K_o; \mathcal{E}(f)) \arrow[ll] \arrow[d, "\approx"] \\
H^m(K'', \partial K''; \mathcal{E}(f,K'')) & & H^m(K, \partial K; \mathcal{E}(f,K)) \\
& H^m(K'', L; \mathcal{E}(f,K'')) \arrow[ul] \arrow[ur, "\approx"'] &
\end{tikzcd}$$

where $L = K'' - K$, and the corresponding diagram where K' replaces K, tells that $o(f,K)$ and $o(f,K')$ coallesce in $H^m(U, U - K''_o; \mathcal{E}(f))$ and hence determine the same element in $H_c^m(U; \mathcal{E}(f))$.

2.8 Propositions. (Homotopy Invariance) Suppose $\Gamma : U \times I \to N$ denotes a homotopy such that $\bigcup_t \operatorname{Fix} \Gamma_t$ is compact; i.e. the homotopy is compactly fixed. Set $\Gamma_o = f$ and $\Gamma_1 = g$. The induced homotopy

$$i \times f \simeq i \times g : U \to N \times N$$

induces a bundle equivalence

$$\begin{tikzcd}
\mathcal{E}(f,U) \arrow[rr, "\bar{\Gamma}"] \arrow[dr] & & \mathcal{E}(g,U) \arrow[dl] \\
& U &
\end{tikzcd}$$

which, in turn, establishes a (coefficient) isomorphism

$$\Gamma^* : H_c^m(U, \mathcal{B}(f,U)) \to H_c^m(U; \mathcal{B}(g,U)) .$$

Then

$$\Gamma^*(o(f)) = o(g) .$$

Proof. Let K denote a compact m-manifold with boundary ∂K such that $\bigcup_t \text{Fix } \Gamma_t \subset \text{int } K$, so that K may be used to determine both $o(f)$ and $o(g)$. The remainder of the proof is standard.

2.9 Theorem. Given $f : U \to M$, then there is a compactly fixed homotopy $\Gamma : U \times I \to M$ such that $H_o = f$ and $H_1 = g$ is fixed point free iff the local obstruction index $o(f) = 0 \in H_c^m(U, \mathcal{B}(f,U))$.

Proof. An immediate consequence of 2.7 and 2.8.

2.10 Remark. Sometimes we will display U in the notation for $o(f)$, i.e., $o(f) = o(f,U)$. Also, if $f : M \to M$ is globally defined $o(f,U)$ will denote $o(f|U)$.

In order to state the "addivity" property of the local index, we recall some facts. Suppose $V_1, V_2, \ldots, V_k$ are mutually disjoint open subsets of the open set U and $C_\ell \subset V_\ell$ are compact subsets. Suppose furthermore, that $\mathcal{G}$ is a local system on U and $\mathcal{G}_\ell = \mathcal{G}|V_\ell$. Then, for each ℓ we have

$$H^m(V_\ell, V_\ell - C_\ell; \mathcal{G}_\ell) \xleftarrow[\approx]{i_\ell^*} H^m(U, U - C_\ell; \mathcal{G}) \xrightarrow{j_\ell^*} H^m(U, U - C; \mathcal{G})$$

where i_ℓ, j_ℓ are inclusions and $C = \bigcup_\ell C_\ell$. The homomorphism $i_\ell^{*-1} j_\ell^*$ induces a homomorphism

$$\alpha_\ell : H_c^m(V_\ell; \mathcal{G}_\ell) \to H_c^m(U; \mathcal{G})$$

and consequently a homomorphism

$$\alpha = \sum \alpha_\ell : \sum_\ell H_c^m(V_\ell; \mathcal{G}_\ell) \to H_c^m(U; \mathcal{G}) .$$

The proof of the following proposition is now a simple exercise.

2.10 <u>Proposition</u> (Additivity) Given $f : U \to M$ (compactly fixed as in 2.3). Suppose $V_1,\ldots,V_k$ are finitely many mutually disjoint open sets such that Fix $f \subset \bigcup_\ell V_\ell$. Let $f_\ell = f|V_\ell : V_\ell \to M$. Then under the homomorphism

$$\alpha : \sum H^m_c(V; \mathcal{B}(f_\ell)) \to H^m_c(U; \mathcal{B}(f))$$

we have

$$\alpha(\sum o(f_\ell,V_\ell)) = o(f,U) \quad .$$

3. Local Nielsen Numbers

In this section we consider maps compactly fixed maps $f : U \to X$, where U is an open set in a Euclidean neighborhood retract (ENR [2]). In particular, then, X may be a manifold (possibly with boundary) or a locally finite polyhedron. Notice that we do not require X to be compact, nor do we requires the map f to be compact. The fact that Fix f is compact is what is essential. We recall also that for ENR's we have a local index theory with the usual properties [2] for maps $f : U \to X$ with compact fixed point set. $I(f,U)$ will denote the index of f on U.

Our objective here is to take a compactly fixed $f : U \to X$ and classify the points of Fix f into <u>local</u> Nielsen classes and develop the necessary elementary properties. Since there is a distinct parallel between the local theory and the well-known global theory [4] we will often omit details.

3.1 <u>Definitions</u>: Let x_o and x_1 denote fixed points of $f : U \to X$. x_o and x_1 are Nielsen equivalent in U proved there is a path C <u>in U</u> from x_o to x_1 such that C and Cf are homotopic with endpoints fixed <u>in X</u>. (Recall that composition of functions is read from left to right.) The resulting equivalence classes are called the local Nielsen classes of f in U. $\mathcal{N}(f,U)$ will denote the set of such classes.

3.2 <u>Proposition</u>. The local Nielsen classes of $f : U \to X$ are finite in number.

<u>Proof</u>: Since X is an ANR, it is ULC ([5]) and this forces each Nielsen class to be open in Fix (f). Since Fix f is compact the result follows.

3.3 <u>Notations</u>. We designate the local Nielsen classes of $f : U \to X$ by $\mathcal{N}(f,U) = \{N_1(f,U), N_2(f,U),\ldots\}$. Furthermore, if $f : X \to X$ is globally defined, we set $N(f,U) = N(f|U, U)$; i.e. a local Nielsen class of $f : X \to X$ on U is taken to be a local Nielsen class of $f|U : U \to X$.

3.4 <u>Definition</u> the <u>index</u> $I(N_j(f,U))$ of a Nielsen class $N_j(f,U)$ is defined to be $I(f,V_j)$ where V_j is an open set in U such that $V_j \cap (\text{Fix } f) = N_j(f,U)$. If the index $I(N_j(f,U)) \neq 0$, we call $N_j(f,U)$ an essential class. Finally, the <u>Nielsen number</u> $n(f,U)$ of $f : U \to X$ is defined to be the number (finite) of <u>essential</u> Nielsen classes.

3.5 (Homotopy Invariance) Theorem. Suppose $H : U \times I \to X$ is a compactly fixed homotopy, i.e. there is a compact set $K \subset U$ such that

$$K \supset \bigcup_t \text{Fix } H_t \quad , \quad 0 \leq t \leq 1 \; .$$

Then, $n(H_o,U) = n(H_1,U)$.

Proof: The proof proceeds in a manner parallel to the proof for compact ANR's in [4]. First, set $f = H_o$ and $g = H_1$ and if C is a path in U set

$$\langle H,C\rangle(t) = H(C(t),t) = H_t(C(t)), \; 0 \leq t \leq 1 \; .$$

Thus, $\langle H,C\rangle$ is a path in X. Now, if $x_o \in \text{Fix } f$ and $x_1 \in \text{Fix } g$, we say that $x_o H x_1$ (x_o is H-related to x_1) provided there exists a C in U from x_o to x_1 with $C \sim \langle H,C\rangle$ (endpoint homotopic) in X. This relation H induces a one-one correspondence $\hat{H}$ from a subset of $N(f,U)$ to a subset of $N(g,U)$ via the relation between Nielsen classes

$$[N(f,U)]H[N(g,U)] \Longleftrightarrow x_o H x_1, \; x_o \in N(f,U), \; x_1 \in N(g,U)$$

(see [4], page 92). Up to this point the fact that the homotopy is compactly fixed is not used. It is used, however, at this point to show that $\hat{H}$ is bijective from the essential Nielsen classes of f to the essential Nielsen classes of g. Because X is locally compact one can assume that the compact set K above contains $\text{Fix } H_t$ in its interior for all t, $0 \leq t \leq 1$. Now, open sets in the interior of K may be used to compute indices of H_t and furthermore $H : K \times I \to X$ may be considered a path in X^K where the compact open topology on X^K coincides with the uniform topology. Now, the proof in [4, pages 93-94] applies to show

a) $[N(f,U)]H[N(g,U)] \Rightarrow I(N(f,U)) = I(N(g,U))$

b) $N(f,U)$ is not H-related to some $N(g,U) \Rightarrow I(f,U) = 0$

this completes the sketch of the proof.

We will also find in useful to express local Nielsen classes in terms of universal covers after the manner of Jiang [6]. Given $f : U \to X$, where X is an ENR, the components of U

are open and since Fix f is assumed compact, Fix f lies in a finite number of these components and each of these components produces distinct local Nielsen classes. There is, therefore, no essential loss of generality if we assume U and X are connected.

Let $\eta : \tilde{X} \to X$, $\eta_U : \tilde{U} \to U$ denote the universal covers of X and U, respectively and $i : U \hookrightarrow X$ the inclusion map. Choose $u_o \in U$, $\tilde{u}_o \in \eta_U^{-1}(u_o)$, $\tilde{x}_0 \in \eta^{-1}(i(u_o))$ $\tilde{y}_o \in \eta^{-1}(f(u_o))$. These choices uniquely determine fixed lifts $\tilde{i}$ and $\tilde{f}$ such that $\tilde{i}(\tilde{u}_o) = \tilde{x}_o$, $\tilde{f}(\tilde{u}_o) = \tilde{y}_o$:

$$\begin{tikzcd} \tilde{U} \arrow[r, "\tilde{i}"] \arrow[d, dash, "\eta_U"'] & \tilde{X} \arrow[d, dash, "\eta"] \\ U \arrow[r, "i"] & X \end{tikzcd} \qquad \begin{tikzcd} \tilde{U} \arrow[r, "\tilde{f}"] \arrow[d, dash, "\eta_U"'] & \tilde{X} \arrow[d, dash, "\eta"] \\ U \arrow[r, "f"] & X \end{tikzcd} .$$

Furthermore, if we let $\pi(U)$ and π denote, respectively, the covering groups of η_U and η, i and f induce homomorphisms

$$i_U : \pi(U) \to \pi \quad , \quad \varphi_U : \pi(U) \to \pi$$

with characterizing equations

$$\sigma\tilde{i} = \tilde{i}i_U(\sigma) \quad , \quad \sigma\tilde{f} = \tilde{f}\varphi_U(\sigma) \quad , \quad \sigma \in \pi(U) \ .$$

We should also note that all the lifts of f have the form $\tilde{f}\alpha$, $\alpha \in \pi$ and $\tilde{f}\alpha = \tilde{f}\beta$ iff $\alpha = \beta$.

Now, let Coin $[\tilde{f}\alpha, \tilde{i}]$ denote the coincidence set of $\tilde{f}\alpha$ and $\tilde{i}$; i.e.

$$\mathrm{Coin}[\tilde{f}\alpha, \tilde{i}] = \{\tilde{u} \in \tilde{U} : (\tilde{f}\alpha)(\tilde{u}) = \tilde{i}(\tilde{u})\} \ .$$

3.6 <u>Proposition</u>. Each set $\eta_U(\mathrm{Coin}[\tilde{f}\alpha, \tilde{i}])$, $\alpha \in \pi$, is a Nielsen class or empty. Furthermore,

$$\eta_U(\mathrm{Coin}[\tilde{f}\alpha, \tilde{i}]) = \eta_U(\mathrm{Coin}[\tilde{f}\beta, \tilde{i}])$$

iff there is a $\sigma \in \pi(U)$ such that

$$\sigma^{-1}\tilde{f}\alpha \ i_U(\sigma) = \tilde{f}\beta$$

or, equivalently, for some $\sigma \in \pi_U$,

$$\varphi_U(\sigma^{-1})\alpha \ i_U(\sigma) = \beta \ .$$

Proof. a) Suppose $\tilde{u}$ and $\tilde{v}$ belong to $\mathrm{Coin}[\tilde{f}\alpha, \tilde{i}]$. Then a path $\tilde{C}$ in $\tilde{U}$ from $\tilde{u}$ to $\tilde{v}$ induces a path C from $u = \eta_U(\tilde{u})$ to $v = \eta_U(\tilde{v})$ in U which does the job for showing that u and v are Nielsen equivalent fixed points in U. Thus,

$$\eta_U(\mathrm{Coin}[\tilde{f}\alpha, \tilde{i}]) \subset \text{some Nielsen class } N(f,U) \quad .$$

b) Each fixed point $u \in U$ determines an $\alpha \in \pi$ as follows. Choose $\tilde{u} \in \eta_U^{-1}(u)$. α is determined by the condition

$$(\tilde{f}\alpha)(\tilde{u}) = \tilde{i}(\tilde{u})$$

so that

$$\tilde{u} \in \mathrm{Coin}[\tilde{f}\alpha, \tilde{i}]$$

and hence

$$u \in \eta_U(\mathrm{Coin}[\tilde{f}\alpha, \tilde{i}])$$

for some $\alpha \in \pi$. It is a simple matter to show that where u and v are Nielsen equivalent in U, we may choose $\tilde{u}$ and $\tilde{v}$ above to yield exactly the same $\alpha \in \pi$. Thus, each local Nielsen class is contained in some $\eta_U(\mathrm{Coin}[\tilde{f}\alpha, \tilde{i}])$. This verifies the first part of Proposition 3.6.

c) Now, suppose

$$\eta_U(\mathrm{Coin}[f\alpha, i]) = \eta_U(\mathrm{Coin}\ [f\beta, i]) \quad .$$

Then, we have $\tilde{u}, \tilde{u}_1$ in $\tilde{U}$ such that

$$(\tilde{f}\alpha)(\tilde{u}) = \tilde{i}(\tilde{u}),\ (\tilde{f}\beta)(\tilde{u}_1) = \tilde{i}(\tilde{u}_1),\ \sigma(\tilde{u}) = \tilde{u}_1,\ \sigma \in \pi(U) \quad .$$

Then

$$(\tilde{f}\alpha)(\tilde{u}) = \tilde{i}(\tilde{u}) \Rightarrow (\sigma^{-1}\tilde{f}\alpha)(\tilde{u}_1) = (\tilde{i}\ i_U(\sigma^{-1}))(\tilde{u}_1)$$

$$\Rightarrow (\sigma^{-1}\tilde{f}\alpha\ i_U(\sigma))(u_1) = \tilde{i}(\tilde{u}_1)$$

$$\Rightarrow \sigma^{-1}(\tilde{f}\alpha)i_U(\sigma) = \tilde{f}\beta \quad , \quad \alpha \in \pi(U) \quad .$$

Since this last equality is equivalent to

$$\varphi_U(\sigma^{-1})\alpha\ i_U(\sigma) = \beta$$

and also implies (as a simple exercise)

$$\eta_U(\mathrm{Coin}[\tilde{f}\alpha,\ \tilde{i}]) = \eta_U(\mathrm{Coin}[\tilde{f}\beta,\ \tilde{i}])\ ,$$

the proof is complete.

3.7 <u>Definition</u>. Given homomorphisms (of groups)

$$\psi : \pi' \to \pi \qquad \phi : \pi' \to \pi\ .$$

We introduce the right action of π' on π by

$$\alpha * \sigma = \varphi(\sigma^{-1})\alpha\ \psi(\sigma) \quad , \quad \sigma \in \pi',\ \alpha \in \pi\ .$$

The resulting set of orbits $R[\psi,\varphi]$ is called the set of Reidemeister classes, i.e. each orbit is a <u>Reidemeister class</u>.

3.8 <u>Definition</u>: Given a compactly fixed $f : U \to X$ and corresponding homomorphisms $i_U : \pi(U) \to \pi$, $\varphi_U : \pi(U) \to \pi$ (as above), we call $R[i_U, \varphi_U]$ the set of local Reidemeister classes on U generated by f.

3.9 <u>Proposition</u>: The correspondence

$$\Gamma : [\alpha] \leftrightarrow \eta_U(\mathrm{Coin}[\tilde{f}\alpha,\ \tilde{i}])$$

takes Reidemeister classes to Nielsen classes bijectively provided we ignore those Reidemeister classes $[\alpha]$ for which $\mathrm{Coin}[\tilde{f}\alpha,\ \tilde{i}] = \phi$.

<u>Proof</u>. Immediate from Proposition 3.6.

Suppose we let $\hat{U}$ denote the component of $\eta^{-1}(U)$ which contains $\tilde{x}_o \in \eta^{-1}(i(u_o))$. Then, $\eta|\hat{U} : \hat{U} \to U$ is a covering map. It is easy to see that $\pi_1(\hat{U}, x_o)$ corresponds to the kernel of $i_U : \pi(U) \to \pi$ and hence the covering map $\eta|\hat{U}$ is regular and furthermore $f : U \to X$ has a unique lift $\hat{f} : (\hat{U}, \tilde{x}_o) \to (\tilde{X}, \tilde{y}_o)$ and hence a diagram

$$\begin{tikzcd}
\tilde{U} \arrow[r, "\tilde{i}"] \arrow[d, "\eta_U"'] & \hat{U} \arrow[r, "\hat{f}"] \arrow[d, "\eta|\hat{U}"] & \tilde{X} \arrow[d, "\eta"] \\
U \arrow[r, "id"] & U \arrow[r, "f"] & X
\end{tikzcd}$$

The following lemma is easy to prove, because $\tilde{i}\hat{f} = \tilde{f}$.

3.10 Lemma. For $\alpha \in \pi$

$$\tilde{i}(\mathrm{Coin}[\tilde{f}\alpha, \tilde{i}]) = \mathrm{Fix}(\hat{f}\alpha)$$

and hence

$$\eta_U(\mathrm{Coin}[\tilde{f}\alpha, \tilde{i}] = \eta|\hat{U}(\mathrm{Fix}\ \hat{f}\alpha) \quad .$$

We also have the following result. Let $\hat{\pi}(U) = \dfrac{\pi(U)}{\ker i_U}$ and $j : \pi(U) \to \hat{\pi}(U)$ the natural projection. We also have diagrams

$$\begin{tikzcd} \pi(U) \arrow[rr, "i_U"] \arrow[dr, "j"'] & & \pi \\ & \hat{\pi}(U) \arrow[ur, "\hat{i}_U"'] & \end{tikzcd} \qquad \begin{tikzcd} \pi(U) \arrow[rr, "\varphi_U"] \arrow[dr, "j"'] & & \pi \\ & \hat{\pi}(U) \arrow[ur, "\hat{\varphi}_U"'] & \end{tikzcd} \quad .$$

3.11 Lemma. Since

$$\varphi_U(\sigma^{-1})\alpha\ i_U(\sigma) = \hat{\varphi}_U(j(\sigma)^{-1})\alpha\ \hat{i}_U(j(\sigma))$$

the identity map $id : \pi \to \pi$ induces a bijection

$$R[i_U, \varphi_U] \xrightarrow{\approx} R[\hat{i}_U, \hat{\varphi}_U] \quad .$$

Thus, Proposition 3.9 may be reformulated as follows:

3.12 Proposition. The correspondence

$$R[\hat{i}_U, \hat{\varphi}_U] \to N(f, U)$$

which takes

$$[\alpha] \mapsto \eta|\hat{U}(\mathrm{Fix}\ (\hat{f}\alpha))$$

takes Reidemeister classes to Nielsen classes bijectively provided we ignore Reidemeister classes $[\alpha]$ for which $\mathrm{Fix}\ (\hat{f}\alpha) = \phi$.

Suppose $U \subset V \subset X$, where U and V are both open, connected subsets of X, $f_V : V \to X$ is a given map, and $\tilde{U}, \tilde{V}, \tilde{X}$ are the corresponding covering spaces. Then, as before we have fixed lifts

$$\tilde{U} \xrightarrow{\tilde{i}_U} \tilde{X},\ \tilde{U} \xrightarrow{\tilde{f}_U} \tilde{X},\ \tilde{V} \xrightarrow{\tilde{i}_V} \tilde{X},\ \tilde{V} \xrightarrow{\tilde{f}_V} \tilde{X}$$

where $\tilde{i}_U$ and $\tilde{i}_V$ cover inclusions (which are not designated) and $\tilde{f}_U$, $\tilde{f}_V$ cover f_V and $f_U = f_V|U$, respectively. Choose the lift $\tilde{i}_U^V : \tilde{U} \to \tilde{V}$ of the inclusion map $U \hookrightarrow V$ with the property that $\tilde{i}_U^V \tilde{i}_V = \tilde{i}_U$. Then, $i_U^V : \pi(U) \to \pi(V)$ is uniquely determined by the condition

$$\sigma \, \tilde{i}_U^V = \tilde{i}_U^V \, i_U^V(\sigma) \quad , \quad \sigma \in \pi(U) \quad .$$

Now a simple argument shows that

$$i_U = i_U^V \, i_V \quad , \quad \varphi_U = i_U^V \, \varphi_V \quad .$$

Furthermore, the identity map $\pi \to \pi$ is equivalent with respect to the map $i_U^V : \pi(U) \to \pi(V)$, thus inducing

$$h_U^V : R[i_U, \varphi_U] \to R[i_V, \varphi_V] \quad .$$

3.13 Convention: If $K \subset X$ is an set and $U = \text{int } K$, it is convenient to set

$$R[i_K, \varphi_K] = R[i_U, \varphi_U], \; \mathcal{N}(f,K) = \mathcal{N}(f,U) \quad .$$

Given a compactly fixed $f : U \to X$, it may be impossible to find a compact set K in U such that the fundamental group $\pi(K)$ "captures" all of $\pi(U)$. Thus, the natural map

$$h_K^U : R[i_K, \varphi_K] \to R[i_U, \varphi_U]$$

need not be injective. However, the following result indicates that such a K captures the essential information on Fix f in U.

3.14 Proposition. Let $f : U \to X$ denote a compact fixed map, where X is an ENR. Then, there exists a compact set $K \subset U$ such that Fix $f \subset$ int K and $\mathcal{N}(f,U) = \mathcal{N}(f,K)$.

Proof. First, using the fact that X is locally compact, choose a compact set L such that Fix $f \subset$ int L. Each Nielsen class $N(f,L)$ of $f|L$ lies in a unique Nielsen class $N(f,U)$, thus defining a surjective function $\psi : \mathcal{N}(f,L) \to \mathcal{N}(f,U)$. If N_i and N_j are Nielsen classes in $\mathcal{N}(f,L)$ such that $\psi(N_i) = \psi(N_j)$, there is a path α_{ij} in U from N_i to N_j such that $\alpha_{ij} \sim f(\alpha_{ij})$. Only finitely many such pairs N_i, N_j occur so that there is a compact set $K \subset U$ such that Fix $f \subset$ int K and the paths α_{ij} are all in int K. Now, it is clear that the corresponding map $\psi : \mathcal{N}(f,K) \to \mathcal{N}(f,U)$ is the identity.

3.15 Corollary. Let $f : U \to M$ denote a compactly fixed map, where U is an open set in the manifold M. Then, there exists a manifold (with boundary) $K \subset U$ such that the Nielsen classes in $N(f,U)$ and $N(f,K)$ correspond identically. Furthermore, if U is connected we may choose K to be connected.

3.16 Corollary. If $f : U \to X$ and K are as in Proposition 3.14, the correspondence

$$h_K^U : R[i_K, \varphi_K] \to R[i_U, \varphi_U]$$

is bijective provided (using Proposition 3.9) we restrict ourselves to Reidemeister classes which correspond to Nielsen classes.

4. Preliminaries to Calculating $o(f,U)$.

Let $p : E \to M \times M$ denote the fiber map (§2) replacing the inclusion map $M \times M - \Delta \subset M \times M$. $F_{(u,v)}$ will denote the fiber over (u,v). Given a tubular neighborhood T of the diagonal $\Delta \subset M \times M$, let $T_o = T - \Delta$. Then, given $u \in M$ and a local orientation of M at we can assign an element

$$g_u \in \pi_{m-1}(F_{(u,v)}) \quad , \quad (u,v) \in T_o$$

as follows: Let $\tilde{\Delta}$ denote the diagonal in $\tilde{M} \times \tilde{M}$, with corresponding tubular neighborhood $\tilde{T}$. If $\tilde{T}_o$ denotes the complement of the 0-section in $\tilde{T}$, we have

$$\begin{tikzcd}
H_m(\tilde{T}, \tilde{T}_o) \arrow[r, "\approx"] \arrow[dr] & H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \tilde{\Delta}) \\
 & H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta)) \arrow[u] \\
 & \pi_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta), (\tilde{u},\tilde{v})) \arrow[u, "\approx"'] \arrow[d, "\approx"] \\
\pi_m(M, M - u, v) \arrow[r, "\approx"] & \pi_m(M \times M, M \times M - \Delta, (u,v)) \arrow[d, "\approx"] \\
 & \pi_{m-1}(F_{(u,v)}, (\bar{u},\bar{v}))
\end{tikzcd}$$

where $(\tilde{u},\tilde{v}) \in \tilde{T}$, $(\tilde{u},\tilde{v}) \mapsto (u,v)$, and $\bar{u}, \bar{v}$ are constant paths at u and v, respectively. The isomorphism $\pi_m(M, M - u, v) \to \pi_m(M \times M, M \times M - \Delta, (u,v))$ is induced by the section $M \to M \times M$ given by $y \mapsto (u,y)$. If we choose a Euclidean neighborhood W of u and an orientation of W, an imbedding

$$i_u : (D^m, S^{m-1}, a_o) \to (W, W - u, v)$$

(which take o to u) determines an element of $\pi_m(M, M - u, v)$ and hence (see the diagram above) an element $g_u \in \pi_{m-1}(F_{(u,v)}, (\bar{u},\bar{v}))$. g_u may be represented in $H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta))$ as follows: Given $\tilde{u}$ over u, the imbedding i_u lifts to an imbedding $\tilde{i}_u : (D^m, S^{m-1}, a_o) \to (\tilde{W}, \tilde{W} - \tilde{u}, \tilde{v})$, where $\tilde{W}$ covers W. Define $\gamma_u : (D^m, S^{m-1}) \to (\tilde{T}, \tilde{T}_o)$ by $\gamma_u(y) = (\tilde{u}, \tilde{i}_u(y))$. $[\gamma_u]$ generates $H_m(T, T_o)$ and determines and element $g_{\tilde{u}} \in H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta))$. If $\tilde{u}\sigma = \tilde{u}_1$, then it is easy to see that $g_{\tilde{u}} = (\operatorname{sgn} \sigma) g_{\tilde{u}_1}$. The following lemma is easy to prove.

4.1 Lemma: Let U denote a connected open set in M. If U is non-orientable, any choice of local orientations leads to a function $g : U \to \mathcal{B}$ with the property that for (x,y) and $(u,v) \in T_o \cap (U \times U)$, there exists a path (α,β) in $T_o \cap (U \times U)$ from (x,y) to (u,v) such that

$$(\alpha,\beta)_\# : \pi_{m-1}(F_{(x,y)}) \to \pi_{m-1}(F_{(u,v)})$$

takes g_x to g_u. In the orientable case the result holds provided local orientations are chosen compatibily.

Now, let (x,y), (u,v), (u',v') belong to $T_o \cap (\text{int } L \times \text{int } L)$ and consider (x,y) as our base point with $\pi_m(F_{(x,y)})$ identified with $\mathbb{Z}[\pi]$, with g_x corresponding to $1 \in \pi$.

4.2 Lemma: Suppose (α,β) is any path from (u,v) to (u',v'). Suppose further that (α_o,β_o), (α_1,β_1) are paths in T_o from (x,y) to (u,v) and from (x,y) to (u',v'), respectively, as in Lemma 4.1 (see diagram below). Then, under the isomorphism of local groups

$$(\alpha,\beta)_\# : \pi_{m-1}(F_{(u,v)}) \to \pi_{m-1}(F_{(u',v')})$$

we have

$$(\alpha,\beta)_\# g_u = (\text{sgn } \sigma)(\alpha_1,\beta_1)_\#(\tau\sigma^{-1})$$

where $(\alpha_o,\beta_o)_\# g_x = g_u$, $(\alpha_1,\beta_1)_\# g_x = g_{u'}$, and $\sigma = \alpha_o\alpha\alpha_1^{-1}$, $\tau = \beta_o\beta\beta_1^{-1}$.

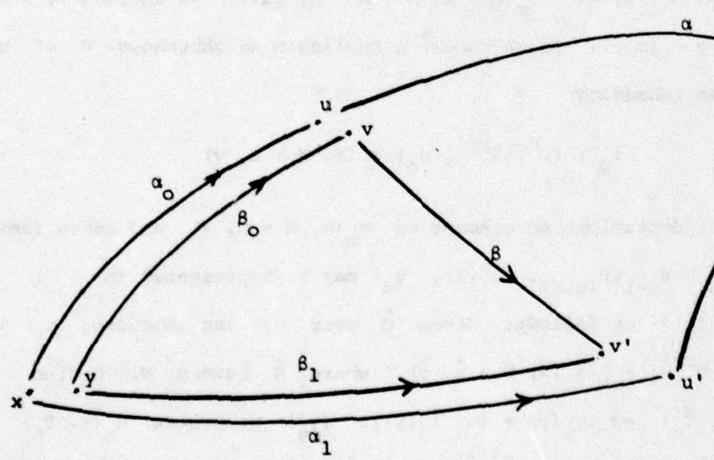

Figure 1

Proof.
$$(\alpha,\beta)_\# g_u = (\alpha,\beta)_\# (\alpha_o,\beta_o)_\# g_x$$
$$= (\alpha_1,\beta_1)_\# (\alpha_1,\beta_1)_\#^{-1} (\alpha,\beta)_\# (\alpha_o,\beta_o)_\# g_x$$
$$= (\alpha_1,\beta_1)_\# [g_x \circ (\sigma,\tau)]$$
$$= (\mathrm{sgn}\ \sigma)(\alpha_1,\beta_1)_\# (\tau\sigma^{-1}) \quad .$$

4.3 *Convention.* If α is a path from u to u' and β is a path from v to v' where u is "close to" v and u' is "close to" v' in the sense that $(u,v) \cup (u',v') \subset T_o$, the statement $\alpha \sim \beta$ (mod endpoints) will mean that there is a homotopy from α to $\beta : H : I \times I \to M$ such that $H(0,t)$ and $H(1,t)$ trace paths, with $(u, H(0,t))$, $(u', H(1,t))$ in T_o. Alternatively, one may replace β by a path β' from u to u' with β' close to β and then $\alpha \sim \beta$ (mod endpoints) mean $\alpha \sim \beta'$ with endpoints fixed, as usual.

4.4 *Corollary.* If in Lemma 4.2, $\alpha \sim \beta$ (mod endpoints), then
$$(\alpha,\beta)_\#(g_u) = (\mathrm{sgn}\ \sigma) g_u ,$$
where $\sigma = \alpha_o \alpha \alpha_1^{-1}$.

Let $f : U \to M$ denote a compactly fixed map with U connected end choose a base point $x_o \notin \mathrm{Fix}\ f$. The local group of $\mathcal{B}(f)$ at x_o is $\pi_{m-1}(F_b)$, where $b = (x_o, f(x_o))$. $\pi_{m-1}(F_b)$ is identified with $\mathbf{Z}[\pi]$ and the right action of $\pi(U) = \pi_1(U, x_o)$ on $\mathbf{Z}[\pi]$ is given by
$$\alpha \circ \sigma = \mathrm{sgn}\ \sigma\ i_U(\sigma^{-1})\alpha\ \varphi_U(\sigma) \quad , \quad \sigma \in \pi(U) \quad , \quad \alpha \in \pi \quad .$$

Define a new right action
$$(*) \qquad \alpha * \sigma = \varphi_U(\sigma^{-1})\alpha\ i_U(\sigma) \quad , \quad \sigma \in \pi(U) \quad , \quad \alpha \in \pi \quad .$$

Now, denote the twisting action of $\pi(U)$ on $\mathbf{Z}$ by
$$n \cdot \sigma = (\mathrm{sgn}\ \sigma) n \qquad \sigma \in \pi(U) \quad , \quad n \in \mathbf{Z}$$
and consider the bilinear pairing
$$P_o : \mathbf{Z}[\pi] \otimes \mathbf{Z} \to \mathbf{Z}[\pi]$$

defined by

$$\alpha \otimes n \mapsto n\alpha^{-1} \ .$$

4.5 Lemma. Let $\sigma \in \pi(U)$, then the pairing (4) satisfies the condition

$$P_o(\alpha \circ \sigma \otimes n \circ \alpha)) = P_o(\alpha \circ n) * \sigma$$

i.e. P_o is equivariant.

Proof.

$$P_o(\alpha \circ \sigma \otimes n \circ \sigma) = P_o(\operatorname{sgn} \sigma \ i_U(\sigma^{-1})\alpha \ \varphi_U(\sigma) \otimes (\operatorname{sgn} \sigma)n)$$

$$= n \ \varphi_U(\sigma^{-1})\alpha^{-1} \ i_U(\sigma)$$

$$= (n\alpha^{-1}) * \sigma$$

$$= P_o(\alpha \circ n) * \sigma \ .$$

Let $\mathcal{T}(U)$ denote the orientation sheaf of twisted integers over U. Then for $x \in U$, the Hurewicz homomorphism

$$h : \pi_m(M, M - x) \to H_m(M, M - x)$$

induces a coefficient homomorphism $h : \mathcal{B}(U) \to \mathcal{T}(U)$ where $\mathcal{B}(U) = \mathcal{B}(i)$ and $i : U \to M$ is inclusion. In particular, using as base point $x_o \in U$, we may identify

$$\pi_{m-1}(F_b) \equiv \mathbb{Z}[\pi] \ \text{ with } \ g_{x_o} \mapsto 1$$

$$H_m(M, M - x) \equiv \mathbb{Z} \ \text{ with } \ h(g_{x_o}) \mapsto 1 \ .$$

4.6 Corollary. Let $\mathcal{R}(f)$ denote the local system on U induced by the action (*). Then, P_o induces a bilinear pairing

$$P : \mathcal{B}(f) \otimes \tau(U) \to \mathcal{R}(f)$$

so that over every $x \in U$

$$P(g_x \otimes h(g_x)) = 1 \ .$$

4.7 Remark. Corollary 4.6 is valid for L a compact connected submanifold with boundary ∂L, $L \subset U$. In particular we have a corresponding pairing

$$P_L : \mathcal{B}(f,L) \otimes \tau(L) \to \mathcal{R}(f,L)$$

where the local systems $\mathcal{B}(f,L)$, $\tau(L)$, $\mathcal{R}(f,L)$ are restrictions from U to L.

Now, let L denote a compact, connected triangulated manifold with boundary ∂L such that $L \subset U$. Assume also that L is triangulated so that adjacent m-simplexes are contained in the same Euclidean neighborhood in U. L determines fundamental classes as follows:

If s is an oriented simplex of L and u_s is a point on ∂s, then using Lemma 4.1, the orientation of s determines an orientation around u_s and thereby an element $g_{u_s} \in \pi_{m-1}(F_b)$, $b = (u_s, v_s)$ and v_s is near u_s. Set $g_s = g_{u_s}$.

4.4 Definition. The m-chain $\sum_s g_s s$, where the sum runs over a basis of oriented m-simplexes of $(L,\partial L)$, determines the homology class

$$\underline{\mu}(L;\ \pi) \in H_m(L,\partial L;\ \mathcal{B}(L))\ ,$$

where $\mathcal{B}(L) = \mathcal{B}(i)$ is induced from $\mathcal{B}$ by $i \times i : L \to M \times M$, which we call the twisted π - fundamental homology class of $(L,\partial L)$ in M.

Let $\underline{\mu}(L) \in H_m(L,\partial L;\ \tau(L))$ denote the classical twisted integral homology class on $(L,\partial L)$ [7]. Since at the chain level $\mu(L)$ has the form $\sum_\sigma h(g_s)s$, one sees that under the induced coefficient homomorphism $h_* : H_m(L,\partial L;\ \mathcal{B}(L)) \to H_m(L,\partial L;\ \mathcal{T}(L))$

$$h_* : \underline{\mu}(L;\ \pi) \mapsto \underline{\mu}(L)\ .$$

The corresponding dual fundamental cohomology is defined as follows:

4.5 Definition. Let s denote an oriented m-simplex of $(L,\partial L)$. The m-cochain

$$c_s(s') = \begin{cases} g_s & \text{if } s' = s \\ 0 & \text{if } s' \neq s \end{cases}$$

leads to a cohomology class $\bar{\mu}(L;\ \pi) \in H^m(L,\partial L;\ \mathcal{B}(L))$ called the twisted π - fundamental cohomology class of $(L,\partial L)$ in M.

4.6 Remark. Using Lemma 4.2 one shows easily that $\bar{\mu}(L; \pi)$ is independent of s, i.e. for $s \neq s'$, c_s and $c_{s'}$ are cohomologous. Also, if we let $\bar{\mu}(L) \in H^m(L, \partial L; \tau(L))$ denote the classical twisted (over $\mathbb{Z}$) cohomology class [7] $\bar{\mu}(L,\pi)$ maps to $\bar{\mu}(L)$, via $h^* : H^m(L, \partial L; \mathcal{B}(L)) \to H^m(L, \partial L; \mathcal{T}(L))$.

4.7 Proposition. $\langle \bar{\mu}(L,\pi), \underline{\mu}(L) \rangle = [1] \in R[i_U, i_U]$.

Proof. Fix a simplex s and a base point $u_s \in \partial s$. Then,

$$c_s(\sum_{s'} h(g_{s'})s') = \Gamma_o(c_s(s) \otimes h(g_s))$$

$$= \Gamma_o(g_s \otimes h(g_s)) \quad .$$

Therefore,

$$\bar{\mu}(L,\pi) \cap \underline{\mu}(L) = [1 \cdot u_s] \in H_o(L; \mathcal{R}(i))$$

where the cap product is induced by the pairing

$$\mathcal{B}(L) \otimes \tau(L) \to \mathcal{B}(L)$$

where $\mathcal{R}(L) = \mathcal{R}(i)$. But, under the isomorphism $H_o(L; \mathcal{R}(L)) \equiv \mathbb{Z}R[i_U, i_U]$, $[1 \cdot u_s]$ corresponds to [1], the Reidemeister class in $R[i_U, i_U]$ containing $1 \in \pi$. Therefore,

$$\langle \bar{\mu}(L,\pi), \underline{\mu}(L) \rangle \equiv \bar{\mu}(L,\pi) \cap \underline{\mu}(L) = [1] \quad .$$

These fundamental classes pass to U is the usual fashion as follows. First, if L_o denotes L minus a small "collar" around the boundary, then the image of $\bar{\mu}(L; \pi)$ under

$$H^m(L, \partial L; \mathcal{B}(L)) \xrightarrow{\approx} H^m(U, U - L_o, \mathcal{B}(L)) \to H^m_c(U; \mathcal{B}(U))$$

determines $\bar{\mu}(U; \pi) \in H^m_c(U; \mathcal{B}(U))$ the twisted π - fundamental cohomology class of U. Furthermore, if $\mathcal{A}$ is the family of compact, connected manifolds L with boundary ∂L such that $L \subset U$, once can choose a compatible $\mathcal{A}$ family [8]

$$\underline{\mu}(U; \pi) = \{\underline{\mu}(L; \pi) \in H_m(L, \partial L; \mathcal{B}(L)) \equiv H_m(U, U - L_o; \mathcal{B}(U))\}$$

and call $\underline{\mu}(U; \pi)$, the twisted π - fundamental homology class of U. In a similar fashion, a compatible $\mathcal{A}$ family

$$\underline{\mu}(U) = \{\underline{\mu}(L) \in H_m(L,\partial L;\ \mathcal{T}(L))\}$$

determines the twisted fundamental class (up to sign) of U.

Finally, for any compactly fixed $f : U \to M$, the pairing

$$P : \mathcal{B}(f) \otimes \mathcal{T}(U) \to \mathcal{R}(f)$$

induces a Kronecker product

$$H_c^m(U;\ \mathcal{B}(f)) \xrightarrow{\langle\cdot,\ \underline{\mu}(U)\rangle} \mathbf{Z}R[i_U, \varphi_U]$$

induced by

$$H^m(L,\partial L;\ \mathcal{B}(f,L)) \xrightarrow{\langle\cdot,\ \underline{\mu}(L)\rangle} \mathbf{Z}R[i_L, \varphi_L] \xrightarrow{h_L^U} \mathbf{Z}R[i_U, \varphi_U]$$

where $\mathcal{B}(f,L)$ is $\mathcal{B}(f)$ restricted to L.

4.8 Remark. A simple direct argument (without invoking duality) shows that

$$\langle\cdot,\ \underline{\mu}(L)\rangle : H^m(L,\partial L;\ \mathcal{B}(f,L)) \to \mathbf{Z}R[i_L, \varphi_L]$$

is an isomorphism.

5. Calculating the Local Obstruction Index $o(f)$

We assume again the data (M, f, U) of 2.3, with the added assumption the U is connected. We also assume that K is a compact manifold with boundary and $\text{Fix } f \subset \text{int } K$. Our immediate objective is to compute the local obstruction index $o(f,K) \in H^m(K,\partial K; \mathcal{B}(f,K))$ of f on K (Definition 2.5). We focus our attention first on one of the components L of K and then $o(f,K)$ will be computed in terms of its components $o(f,L) \in H^m(L,\partial L; \mathcal{B}(f,L))$. Thus our immediate objective is to prove, using the notation in §4, the following result.

5.1 Theorem. Suppose $f : U \to M$ is a compactly fixed map and L a connected compact submanifold with boundary ∂L such that $L \subset U$ and $(\text{Fix } f) \cap \partial L = \phi$. If $o(f,L)$ is the local obstruction index of f on L in U, then using the pairing (§4) $\mathcal{B}(f,L) \otimes \mathcal{T}(L) \to \mathcal{R}(f,L)$ we have that under the isomorphism

$$\langle \cdot, \underline{\mu}(L) \rangle : H^m(L,\partial L; \mathcal{B}(f,L)) \to \mathbb{Z}R[i_L, \varphi_L]$$

$$\langle o(f,L), \underline{\mu}(L) \rangle = \sum_{\rho \in R} I(\rho)\rho$$

where $R = R[i_L, \varphi_L]$ is the set of Reidemeister classes and $I(\rho)$ is the index of the Nielsen class corresponding to ρ under the map $\Gamma : R[i_L, \varphi_L] \to \mathcal{N}(f,L)$ of Proposition 3.9.

Before, giving the proof of Theorem 5.1, we prove a succession of lemmas. Some of these closely parallel corresponding ones in the global case [1] so we may omit some details.

We assume now (without loss of generality), in addition to the previous data that $\text{Fix }(f) \cap L$ is finite and each fixed point lies in the interior of a maximal simplex of a triangulation of L. Furthermore, each such simplex s is contained in a Euclidean neighborhood V_s and if $\text{Fix}(f) \cap s \neq \phi$, then $f(s) \subset V_s$.

Consider the section $u = u_L : L - \text{Fix } f \to E^*$ given by

$$u(y) = (\bar{y}, \overline{f(y)} \qquad x \in L - \text{Fix } f \ .$$

Thus, the cochain $c(f,L) \in C^m(L,\partial L; \mathcal{B}(f,L))$, representing the obstruction $o(f,L)$ is given by the following:

If s is an oriented m-cell, then

$$c(f,L)(s) = \begin{cases} 0 & \text{if } s \cap \text{Fix } f = \phi \\ [\varphi_s] \in \pi_{m-1}(q^{-1}(u_s)) & \text{otherwise} \end{cases}$$

where $u_s \in \partial s$ and when (D^m, S^{m-1}, a_o) and $(s, \partial s, u_s)$ are identified, preserving orientations,

$$\varphi_s(u) = (\overrightarrow{u\,u_s}, \overrightarrow{(f(u)\ f(v_s))})$$

where $\overrightarrow{uv}$ is the directed line segment from u to v (see Figure 2).

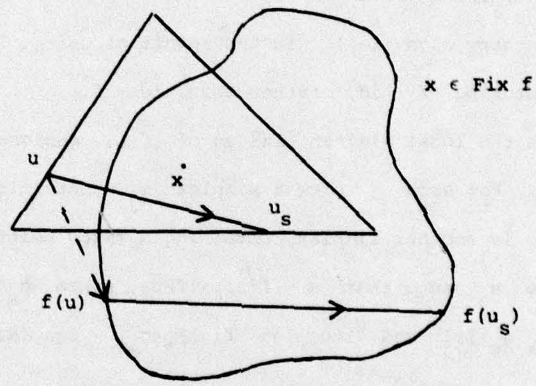

Figure 2.

Then, as noted in [1], a simple homotopy argument shows that if we let $\delta_s : (x,\partial s) \to (M, M - x)$ be given by

$$\delta_s(u) = f(u) - u$$

where $V_s \equiv R^m$ and $x \equiv 0$, $x \in \text{Fix } f \cap (\text{int } s)$, then if we let $\gamma_s = \delta_s + u_s$ (translation by u_s), we have

$$c(f,L)(s) = \begin{cases} 0 & \text{if } s \cap \text{Fix } f = \phi \\ [\gamma_s] & \text{, otherwise} \end{cases} .$$

where $[\gamma_s] \in \pi_m(M, M - u_s, f(u_s)) \cong \pi_m(M \times M, M \times M - \Delta, (u_s, f(u_s))) \cong \pi_{m-1}(F_{(u_s, f(u_s))})$. Thus, since $u - f(u)$ determines the (numerical) *local index* $I(f,x)$ at x, we have

$$c(f,L)(s) = \begin{cases} 0 & \text{if } s \cap \text{Fix } f = \phi \\ (-1)^m \operatorname{Ind}(f,x) g_\sigma, & \text{otherwise} \end{cases} .$$

Thus, we have the following proposition.

5.2 Lemma. The local obstruction index $o(f,L)$ has the cochain representation

$$c(f,L) = (-1)^m \sum_s [I(f,s) g_s] s$$

where $I(f,s)$ is the local index of f on s.

5.3 Remark. The unhappy sign $(-1)^m$ is the result of using $1 \times f : U \to M \times M$, rather than $f \times 1$; thus encountering $f - \mathrm{id}$, rather than $\mathrm{id} - f$.

Let $N(f,L)$, denote the local Nielsen classes of $f|L$, designated individually by $N_1(f,L), \ldots, N_j(f,L), \ldots$. For each j pick a simplex s_j containing a fixed point representing $N_j(f,L)$. If s is another simplex containing a fixed point of $N_j(f,L)$, then there is a path α from s to s_j such that $\alpha \sim f(\alpha)$. Thus, since $g_s s$ to cohomologus to $[\operatorname{sgn}(\alpha, s, s_j)(\alpha, f(\alpha))_\#(g_s)] s_j$ and since $(\alpha, f(\alpha))_\#(g_s) = \operatorname{sgn}(\alpha, s, s_j) g_{s_j}$, we have

5.4 Proposition. The local obstruction index $o(f,L)$ has the cochain representation

$$c'(f,L) = (-1)^m \sum_j [I(N_j(f,L)) g_{s_j}] s_j$$

where the sum is over the local Nielsen classes $N(f,L)$ and $I(N_j(f,L))$ is the (numerical) index of $N_j(f,L)$.

5.5 Corollary. (Local Wecken Theorem). A necessary and sufficient condition that $f|L$ be deformable in M (relative to ∂L) to a fixed point free map is that the local Nielsen number $n(f,L) = 0$, i.e. $n(f,L) = 0 \Longleftrightarrow o(f,L) = 0$.

Now, choose a simplex s_1 in L and assume that our base point is $u_1 \in \partial s_1$ and we identify $\pi_{m-1}(F_{(u_1, f(u_1))})$ with $\mathbb{Z}[\pi]$, g_{s_1} corresponding to 1. See Figure 3.

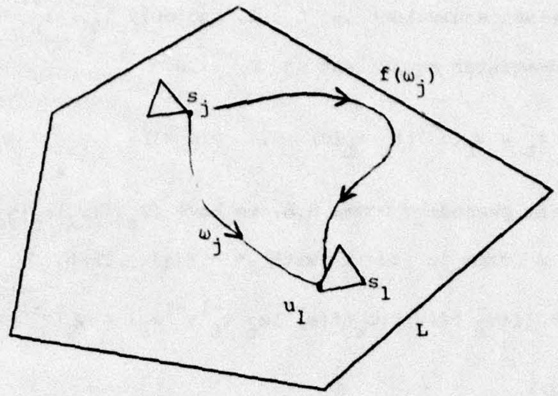

Figure 3.

Choose for each j, a path ω_j in L such that

$$(\omega_j,\omega_j)_\#(g_{s_j}) = g_{s_1} \quad .$$

Then, $g_{s_j} s_j$ is cohomologus to $[\mathrm{sgn}\,(\omega_j,s_j,s)(\omega_j,f(\omega_j))_\#(g_{s_j})]s_1$ where, by Lemma 4.2,

$$(\omega_j,f(\omega_j))_\#(g_{s_j}) = \mathrm{sgn}\,\sigma_j(\tau_j\sigma_j^{-1})$$

where $\sigma_j = [\omega_j^{-1}\omega_j]$, $\tau_j = [\omega_j^{-1}f(\omega_j)]$. Since $\sigma = 1$ and $\mathrm{sgn}(\omega_j,s_j,s) = 1$, we have $g_{s_j}s_j$ cohomologous to $\tau_j s_1$ where $\tau_j = [\omega_j^{-1}f(\omega_j)]$. See Figure 3.

5.6 Lemma. *The local obstruction index* $o(f,L)$ has the following cochain representation concentrated at s_1 where the local group at s_1 is identified with $\mathbb{Z}[\pi]$:

$$c''(f,L) = (-1)^m\,(\sum_j I(N_j(f,L))\tau_j)s_1$$

where $\tau_j \in \pi$ is given by $\tau_j = [\omega_j^{-1}f(\omega_j)]$ for an appropriate path ω_j from the Nielsen class $N_j(f,L)$ to the Nielsen class $N_1(f,L)$.

5.7 Lemma. If x_s and x_t are fixed points of $f|L$ in simplexes s and t, respectively and if ω_s, and ω_t are paths from s to s_1 and t to s_1 such that

$$(\omega_s,\omega_s)_\# g_s = g_{s_1},\ (\omega_t,\omega_t)_\# g_t = g_{s_1}$$

then x_s and x_t are Nielsen equivalent in L if, and only if, $\tau_s^{-1} = [f(\omega_s^{-1})\omega_s]$ and $\tau_t^{-1} = [f(\omega_t^{-1})\omega_t]$ are Reidemeister equivalent on L, i.e.

$$\tau_s = \varphi_L(\sigma^{-1})\tau_t \, i_L(\sigma) \quad , \quad \sigma \in \pi(L) \quad .$$

Proof. By the argument preceding Lemma 5.6, we have $(\omega_s, f(\omega_s))_{\#}(g_s s) = \tau_s = [\omega_s^{-1} f(\omega_s)]$. Suppose γ is a path in L from s to t with $\gamma \sim f(\gamma)$. Then,

$$\tau_s^{-1} = [f(\omega_s^{-1})\omega_s] = [f(\omega_s^{-1})f(\gamma)f(\omega_t)f(\omega_t^{-1})\omega_t \, \omega_t^{-1}\gamma^{-1}\omega_s] = \varphi_L(\sigma^{-1})\tau_t \, i_L(\sigma) \quad ,$$

where $\sigma = [\omega_t^{-1}\gamma^{-1}\omega_s] \in \pi(L)$.

5.8 Lemma. Let Γ denote the correspondence of Proposition 3.9 from the Reidemeister classes $R[i_L, \varphi_L]$ to the Nielsen classes $N(f,L)$. Then, if $\tau_j = [\omega_j^{-1} f(\omega_j)]$, as in Proposition 5.6, we have

$$\Gamma([\tau_j^{-1}]) = N_j(f,L)$$

Proof. Let x_j denote the fixed point in s_j, and x_1 the fixed point in x_1. Use x_1 as base point and then apply part b) of the proof of Proposition 3.6.

If $\Gamma : R[i_L, \varphi_L] \to N(f,L)$ is the correspondence of Proposition 3.9, between Reidemeister classes and Nielsen classes, then we set $N_\rho = \Gamma(\rho)$. Also, we set $I(\rho) = I(N_\rho)$, the index of the corresponding Nielsen class. Of course, if $\Gamma(\rho) = \phi$, we set $I(\rho) = 0$.

We can now give a short proof of Theorem 5.1.

Proof of Theorem 5.1. Using Lemma 5.6

$$\langle c''(f,L), \sum_s h(g_s)s \rangle = (-1)^m \sum_j I(N_j(f,L))\tau_j^{-1} \in \mathbb{Z}[\pi] \quad .$$

Passing to Reidemeister classes on the right, we obtain

$$\langle o(f,L), \underline{\mu}(L) \rangle = (-1)^m \sum_\rho I(\rho)\rho \in \mathbb{Z}R[i_L, \varphi_L] \quad .$$

5.9 Corollary. Let $f : U \to M$ be compactly fixed and let $K = \amalg L_j$, a finite disjoint union of connected submanifolds with boundary. Then under the isomorphism

$$\sum_j \langle \cdot, \underline{\mu}(L_j) \rangle : H^m(K, \partial K; \mathcal{B}(f,K)) \equiv \sum_j H^m(L_j, \partial L_j; \mathcal{B}(f,L_j))$$

$$\downarrow$$

$$\mathbf{Z}R[i_K, \varphi_K] \equiv \sum_j \mathbf{Z}R[i_{L_j}, \varphi_{L_j}]$$

we have

$$\langle o(f,K), \sum \underline{\mu}(L_j) \rangle = \sum_j \sum_{\rho \in R_j} I(\rho)\rho$$

where $R_j = R[i_{L_j}, \varphi_{L_j}]$.

5.10 Corollary (global case). Let $f : M \to M$ denote a self map of a compact, connected manifold with boundary ∂M such that Fix f is compact and (Fix f) $\cap \partial M = \phi$. Then the global obstruction index

$$o(f) \in H^m(M; \mathcal{B}(f))$$

is given by

$$\langle o(f), \underline{\mu}(M) \rangle = \sum_{\rho \in R} I(\rho)\rho$$

where $R = R[id, \varphi]$ and $\varphi = f_*: \pi \to \pi = \pi_1(M)$.

5.11 Corollary. Let $f : U \to M$ be compactly fixed. Suppose K is a compact submanifold with boundary such that $K \subset U$, Fix $f \subset$ int K and the Nielsen classes $\mathcal{N}(f,U)$ and $\mathcal{N}(f,K)$ are identical. (The existence of such a K is guaranteed by Proposition 3.14). Then $o(f) = 0$ if, and only if, $o(f,K) = 0$.

Proof. The "if part" is obvious. On the other hand suppose $o(f) = 0$. Then for some K', $K \subset K' \subset U$ we have $o(f,K') = 0$ and hence

$$0 = \langle o(f,K'), \underline{\mu}(K') \rangle = \sum_{\rho \in R'} I(\rho)\rho$$

thus, $I(\rho) = 0$ for all Reidemeister classes in $R' = R[i_{K'}, \varphi_{K'}]$. Consequently, all the Nielsen classes in K' have index 0. This forces all the Nielsen classes of f relative to K to be inessential and thus

$$\langle o(f,K),\ \underline{\mu}(K)\rangle = \sum_{\rho\in R} I(\rho)\rho = 0$$

therefore, $o(f,K) = 0$.

5.12 Theorem. Suppose $f : U \to M$ is compactly fixed with U connected. Then, under the isomorphism

$$\langle\cdot,\ \underline{\mu}(U)\rangle : H_c^m(U;\ \mathcal{B}(f)) \to \mathbb{Z}R[i_U, \varphi_L]$$

we have

$$\langle o(f),\ \underline{\mu}(U)\rangle = \sum_{\rho\in R} I(\rho)\rho$$

where $R = R[i_U, \varphi_U]$.

Proof. Choose a connected K satisfying the condition of Corollary 5.11. Let

$$h_K^U : R' = R[i_K, \varphi_K] \to R[i_U, \varphi_U] = R$$

denote the correspondence in §3. Then,

$$\langle o(f),\ \underline{\mu}(U)\rangle = h_K^U\langle o(f,K),\ \underline{\mu}(K)\rangle = h_K^U\ (\sum_{\rho\in R'} I(\rho)\rho) = \sum_{\rho\in R} I(\rho)\rho \quad .$$

5.13 Corollary. Suppose $f : U \to M$ is compactly fixed. Then f is deformable, via a compactly fixed homotopy, to a fixed point free map $g : U \to M$ if, and only if, the local Nielsen number $n(f,U) = 0$.

Suppose now that $f : U \to M$ as usual, $L = \coprod_j L_j \subset K \subset U$ such that $\mathrm{Fix}\ f \subset \coprod_j (\mathrm{int}\ L_j)$, and L_j, K are connected submanifolds with boundary. We want to describe now how $o(f,L)$ in $H^m(L,\partial L;\ \mathcal{B}(f,L))$ "coallesces" to $o(f,K)$ in $H^m(K,\partial K;\ \mathcal{B}(f,K))$ thus yielding the appropriate "additivity property" for our generalized local index. We make use of the correspondences (§3)

$$h_{L_j}^K : R[i_{L_j}, \varphi_{L_j}] \to R[i_K, \varphi_K] \quad .$$

5.14 Lemma. If $\rho \in R[i_K, \varphi_K]$ and $I(\rho)$ is its (numerical index, then

$$I(\rho) = \sum_j \sum_{\beta \in P_j} I(\beta)$$

where $P_j = \{\beta : h^K_{L_j}(\beta) = \rho\}$.

Proof. Let $N_\beta(L_j,f)$ denote the Nielsen class in $\mathcal{N}(L_j,f)$ corresponding to $\beta \in P_j$, and $N(\rho)$ the Nielsen class in $\mathcal{N}(K,f)$ corresponding to ρ. It suffices to prove that

$$\coprod_j \coprod_{\beta \in P_j} N_\beta(L_j,f) = N(\gamma) \quad .$$

Recall (§3) that given a fixed point $x \in N(\rho)$, the Reidemeister class ρ is determined by the element $\alpha \in \pi$ subject to the condition

$$(\tilde{f}_K\alpha)(\tilde{x}) = \tilde{i}_K(\tilde{x})$$

where $\tilde{x} \in \eta_K^{-1}(x)$. Such an x belongs to some L_j and hence to some Nielsen class $N_\beta(L_j,f)$ where β is the L_j-Reidemeister class belonging to $N_\beta(L_j,f)$. We need to show that $h^K_{L_j}(\beta) = \rho$. Or, equivalently that α also represents β. Choose $\tilde{x} = \tilde{i}^K_{L_j}(\tilde{y})$ and then

$$(\tilde{f}_L\alpha)(\tilde{y}) = (\tilde{i}^K_{L_j}\tilde{f}_K\alpha)\tilde{y} = \tilde{i}_L(\tilde{y})$$

and thus α does represent β, and hence

$$N(\gamma) \subset \coprod_j \coprod_{\beta \in P_j} N_\beta(L_j,f) \quad .$$

The reverse inclusion has a similar argument and is omitted.

The following theorem is a consequence of Lemma 5.14.

5.15 Theorem (Additivity). Let $f : U \to M$ be compactly fixed and suppose $V = \coprod_j V_j$ is a disjoint union of open sets in U covering Fix f. We identify

$$o(f,U) \equiv \sum_{\rho \in R} I(\rho)\rho \quad , \quad o(f,V_j) \equiv \sum_{\beta \in R_j} I(\beta)\beta$$

where $R = R[i_U, \varphi_U]$, $R_j = R[i_{V_j}, \varphi_{V_j}]$. Then, under the correspondence

$$h_V^U : R[i_V, \varphi_V] \to R[i_U, \varphi_U]$$

$$o(f,V) \equiv \sum_j \sum_{\beta \in R_j} I(\beta)\beta \to \sum_{\rho \in R} (\sum_j \sum_{\beta \in P_j(\rho)} I(\beta))\rho \equiv o(f,U)$$

where $P_j(\rho) = \{\beta : | h_{V_j}^U(\beta) = \rho\}$.

5.16 Remark. When M is 1-connected, Theorem 5.15 reduces to

$$I(f,K) = \sum_j I(f,L_j)$$

the "addivity property" of the classical (numerical) local index.

The next result is another application of Theorem 5.1.

5.17 Theorem. Suppose $f : M \to M$ is a compactly fixed map on a connected manifold with boundary such that $(\text{Fix } f) \cap \partial M = \phi$. Suppose K is a connected submanifold with boundary and $\text{Fix } f \subset \text{int } K$. If $i_K : \pi(K) \to \pi$ is surjective then

a) $h_K^M : R[i_K, \varphi_K] \to R[i_M, \varphi_M]$ is bijective

b) $N(f,K) \equiv N(f,M) = N(f)$

c) $n(f,K) = n(f,M) = n(f)$

d) $o(f,K) = 0$ if, and only if, $o(f,M) = 0$.

Proof. a) is a simple exercise which establishes a one-one correspondence between Nielsen classes relative to K and Nielsen classes relative to M. Then d) is an immediate consequence of Theorem 5.1.

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Given a compactly fixed map $f:U \to M$, where U is an open subset of a manifold M, it is a classical result that one can assign an integer-valued index $I(f,U)$ to this situation with the property that $I(f,U) \neq 0$ implies f (and any compactly fixed perturbation of f) has fixed points in U, i.e. solutions to the equations $f(x) = x$. However, it can still happen that $I(f,U) = 0$ and f has essential fixed points in U. The objective of this paper is to provide a finer invariant $o(f,U)$, called the local obstruction index, which has the property that $o(f,U) \neq 0$ if, and only if, every compactly fixed perturbation

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ABSTRACT (continued)

of f has fixed points in U. $o(f,U)$ is not integer-valued but takes its value initially in the cohomology group $H_c^m(U,\mathcal{B}(f))$ where $m = \dim M$, and $\mathcal{B}(f)$ is an appropriate local coefficient system on U with local group $\mathbb{Z}[\pi]$, $\pi = \pi_1(M)$. In order to compute $o(f,U)$ one employs a Kronecker product

$$(\ \cdot\ ,\mu) : H_c^m(U;\mathcal{B}(f)) \to \mathbb{Z}R[i_U,\varphi_U]\ .$$

$R[i_U,\varphi_U]$ is the set of Reidemeister classes obtained from the group π using the Reidemeister action

$$\alpha*\sigma = \varphi_U(\sigma^{-1})\alpha\ i_U(\sigma)\ ,\quad \sigma \in \pi(U) = \pi_1(U),\ \alpha \in \pi$$

where $i_U : \pi(U) \to \pi$ is induced by inclusion, $\varphi_U = f_* : \pi(U) \to \pi$, and μ is the twisted fundamental class on U. Then, we have the following basic result:

<u>Theorem</u>. $\langle o(f,U),\mu\rangle = \sum_{\rho\in R} I(f,\rho)\rho$

where $R = R[i_U,\varphi_U]$, and $I(f,\rho)$ is the classical integer-valued index of f on the Nielsen class corresponding to the Reidemeister class ρ.